Statistical-Discrete-Gust Method for Predicting Aircraft Loads and Dynamic Response

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The statistical-discrete-gust (SDG) method comprises a stochastic model for atmospheric turbulence and an associated technique for predicting the statistics of aircraft response. In the stochastic model, localized patterns of fluctuation are represented in terms of discrete ramp elements, and associated probability distributions are defined. With appropriate parameter settings, the model may be used to represent either continuous turbulence or relatively isolated gusts. The statistics of response, which may be linear or nonlinear, are derived by an application of the Laplace asymptotic approximation and are expressed in a form that shows the dominant influence of a particular tuned, or worst-case, gust pattern which is dependent on the aircraft dynamics. This paper reviews the basic concepts of the theory with particular reference to the incorporation of fractal representations of turbulence intermittency and to the problem of predicting the effects of nonlinear aircraft dynamics. The determination of numerical parameters from measured data is discussed, both for continuous turbulence and relatively isolated gusts.

Nomenclature

a = intermittency parameter, Eqs. (6) and (11)
$a(H)$ = scale-dependent intermittency parameter, Eq. (20)
b = intensity parameter, Eqs. (6) and (11)
$\bar{c}$ = mean wing chord
D = fractal dimension of region of active turbulence
d = gust gradient distance in airworthiness regulations
F = function defining probability distribution, Eqs. (6) and (11)
f = frequency
f_0 = signal center frequency
H = gust gradient distance for isolated ramp-gust, overall size of gust pattern, Eq. (9)
$\bar{H}$ = tuned gust length
H_i = gradient distance of component ramp gust
H_n = configuration vector defining shape of n-component gust pattern
H_n^* = class of equivalent gust patterns having given shape but different sizes
$\bar{H}_m(y)$ = value of H associated with threshold level y
$\bar{H}_m(u_k)$ = value of H associated with intensity variable u_k
$\bar{h}$ = range of gust gradient distances
J_i = distance variables defining relative positions of ramp-gust components
k = similarity parameter
L = scale length of turbulence
m = value of n for tuned gust pattern
n = number of ramp elements in gust pattern
N_{H,u_k} = average rate of occurrence of gusts, Eq. (6)
N_{H_n,u_k} = average rate of occurrence of n-component gust patterns, Eq. (11)
n_y = average rate of occurrence of response peaks exceeding threshold y
$n_{H,y}$ = subset of n_y associated with gusts of approximate length H
$n_{H_n,y}$ = subset of n_y associated with gust patterns with configuration close to H_n
$p[H_n^*]$ = pattern-amplitude factor, Eq. (12)
s = distance
t = independent variable in function $x(t)$, time
u_k = gust intensity variable, Eqs. (5) and (12)
$u_k(H_n,y)$ = value of u_k required for pattern H_n to produce response peak of amplitude y
$u_k(y)_m$ = minimum value of $u_k(H_n,y)$
$v(s)$ = turbulence velocity component
w = isolated ramp-gust amplitude
w_i = amplitude of component ramp gust
w_g = gust amplitude in airworthiness regulations
$x(t)$ = signal profile
y = threshold amplitude for response peaks
$y(H_n,u_k)$ = peak amplitude in response associated with input pattern (H_n,u_k)
$y(u_k)_m$ = maximum value of $y(H_n,u_k)$
α = $0.89a$
β = $b/1.12$, ratio of volumes of active turbulence in β-model at successive scales
$\gamma(H)$ = amplitude of response to isolated ramp-gust input with gradient distance H
$\bar{\gamma}_m$ = maximum amplitude of response associated with constraint $u_k = 1$
Δf = signal bandwidth
Δt = signal duration
λ = gust-length sensitivity factor
σ_i = sign variable for ramp-gust components
$\psi(H)$ = distance of response peak from onset of gust with gradient distance H
$\bar{\psi}$ = tuned value of $\psi(H)$

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I. Introduction

ALTERNATIVE theoretical descriptions of the turbulent wind developed for the prediction of the response of aircraft have been reviewed by Etkin.[1] He makes the point that our models of the wind have to accommodate those events that are perceived as discrete, and usually described as gusts, as well as the phenomenon described as continuous turbulence, "even though some 'discrete' gusts are actually rare excursions of a continuous process."[1]

Etkin goes on to outline current models for discrete gusts and random turbulence. He describes discrete events as isolated encounters with steep gradients in the speed of the air, typically occurring at the edges of thermals and downdraft, in wakes of mountains, etc., or at temperature inversions. They may also appear as rare extremes of turbulence in clouds, storms, etc., possibly associated with organized structures embedded in the otherwise random background. These organized extremes are not adequately allowed for in the usual Gaussian models of continuous random turbulence.[1]

Nevertheless, the statistical nature of turbulence has in the past usually been treated by means of power spectral density (PSD) methods, based on the theory of stationary random processes. Particular use is made of the formula, originally due to Rice,[2] for threshold exceedance rates, applicable when the aircraft response is evaluated for the case of a Gaussian-process input. Such methods are traditionally employed to meet airworthiness requirements for aircraft flight in "continuous turbulence," although it is generally accepted[1] that, even in continuous turbulence, the more extreme fluctuations are not adequately represented by Gaussian models.

To take some account of the more extreme fluctuations, discrete gust models have been employed. These traditionally have been defined by a specified profile, usually of (1 cos) from (either a complete or half-cycle). U.S. Federal Air Regulations fix the gradient distance as $2d = 25\bar{c}$ and specify values of w_g corresponding to prescribed points of the flight envelope. British regulations, on the other hand, have required that the d be varied to find peak response.

More recently, the statistical-discrete-gust (SDG) model has been proposed,[3-6] in which the variation of w_g with d is related explicitly to probability, and a procedure for evaluating the associated aircraft response, based on a worst-case analysis to find the associated tuned gust or gust pattern, is described which enables aircraft response to be derived in statistical terms.

The SDG model was developed[3-6] primarily as a means of modeling the non-Gaussian characteristics of inertial-range turbulence and the associated effects on the response of aircraft. The motivation for developing an alternative to the widely used PSD theory lies primarily in the failure of the latter to take adequate account of the strong statistical correlations that exist between the phases of Fourier components in turbulence velocity. The PSD carries no phase information, and, to predict the probability distribution of the response, the PSD method introduces the additional assumption that phases are purely random and hence that local patches of turbulence can be represented as samples of a Gaussian process. In fact, the phases are strongly correlated, leading to the existence, even within a field of continuous turbulence, of localized fluctuations characterized by large velocity differences (discrete gusts), significantly more intense that would be the case if the phases were random. Such fluctuations are, in turn, strongly correlated with large peaks in aircraft response.

The purpose of this paper is to describe the basic principles and current status of the SDG method. Particular reference is made to the use of fractals in the representation of turbulence intermittency and to the effects of nonlinear aircraft dynamics on response statistics. The determination of numerical parameters from measured data is described, both for samples of continuous turbulence obtained using specially instrumented aircraft and for relatively isolated gusts encountered in routine operational flying.

II. Theoretical Background

The SDG method was first presented[3] in 1968 in a primitive form applicable only to single isolated gusts. Later it was developed[4-6] by the introduction of more general gust patterns, each modeled as a number of discrete-ramp-shaped components. It was shown how probability distributions could be attached to such patterns and the associated statistics of aircraft response could be derived. The basic capability of the method as a tool for predicting the effects of aircraft dynamics in response to non-Gaussian turbulence was subsequently demonstrated[7,8] by comparison with the results of numerical simulation using inputs consisting of measured samples of turbulence. Also, predictions made using the SDG method have been compared[9] with those based on the PSD method showing that, when the parameters in the SDG method that define the relative probabilities of gust patterns of different shapes are specifically chosen to take the values appropriate to a Gaussian process, an equivalence is obtained. However, it has also been shown,[10] by an analysis of measured gust encounters during operational flying, that at the more severe levels of turbulence a systematic trend of variation of these parameters occurs, associated with a tendency for the gusts to become relatively more isolated. This leads to an associated divergence between the predictions of the SDG and PSD methods.

The PSD method is based on the well-known tools of stationary random process theory. In contrast, when it first appeared, the SDG method suffered from the relative disadvantage that it was based on mathematical concepts that were unfamiliar. However, the relevant mathematical tools, particularly fractals and the use of integrals over a space of variable patterns, are now more widely used, at least within the statistical physics community, than they were 20 years ago. In the following we refer to some of these developments.

The earliest formulation[1] of the SDG method took the concept of self-similarity as a basic hypothesis for modeling turbulence and drew on work by Mandelbrot[11,12] for this idea. However, it was only later[13] that Mandelbrot coined the expression "fractal" to express and generalize the concept and described applications to a wide range of geophysical phenomena, including turbulence. As we shall illustrate, such subsequent generalizations allow the SDG method to be extended beyond the constraints imposed by the assumption of simple self-similarity. In particular, recent ideas in fluid mechanics[14,15] regarding the fractal structure of intermittency in turbulence can be incorporated and their implications for aircraft response can be evaluated (Sec. V).

Another mathematical concept that is fundamental to the formulation of the SDG method is that of a probability distribution over a space of variable patterns. In the SDG model, probability distributions are explicitly attached to patterns of fluctuation occurring in the turbulence input. There results an expression for threshold-exceedance rates for system response in the form of an integral over all possible input patterns, each weighted by its associated probability. Using a standard technique, the Laplace method, this integral can be evaluated asymptotically for large values of the response.

In a recent paper,[16] it was shown that probability distributions may also be attached to particular realizations of a Gaussian process defined by its PSD. In addition, it has been shown that, on this basis, the standard equations of Rice,[2] which form the basis of the PSD method, can be reformulated as a variational problem that calls for a worst-case analysis of system response using deterministic inputs subject to generalized energy constraints that depend on the PSD of the input. This reformulation of Rice's equations provides an overlap of common problems in which the SDG and PSD methods give equivalent results.[9,17]

A close relationship exists between the mathematical methods used in the SDG method and methods used in statistical physics, where integrals over spaces of microscopic configurations occur in the expression for the partition function. This area of commonality has significance when we turn to the problem of nonlinear aircraft response. In particular, the phenomenon of a phase transition, a fundamental concept in statistical physics,[18] appears as an explicit consequence of nonlinearity when the problem is formulated in terms of such integrals. In evaluating the effects of aircraft dynamic nonlinearity using the SDG method, such a phase transition can appear as a sharp amplitude-dependent discontinuity in the statistics of aircraft response to turbulence that may not easily be detectable using existing engineering methods (Sec. VI).

III. Representation of Turbulence

One-Dimensional Model

The excitation of the dynamic response of an aircraft flying through turbulence arises from fluctuations in air velocity that may be resolved into vertical, lateral, and longitudinal components, correlated with one another and varying over the aircraft surface. In the representation generally employed by aeronautical engineers, the turbulence field is defined as a function of position in space and a "frozen-field" hypothesis is invoked to convert this to a function of time as sensed by the aircraft. A further simplification made in common engineering practice that is employed in applications of the SDG method is to consider the aircraft motion to take the form of perturbations from steady rectilinear flight.

In many cases the response of a particular aircraft state variable is predominantly associated with a particular turbulence component whose variation over the span of the aircraft may be neglected to a first approximation. For example, at high speeds the acceleration of the aircraft center of gravity in a direction normal to the aircraft may be associated with the vertical component of turbulence on the aircraft centerline. Thus, we arrive at a very simplified one-dimensional representation that, of course, has its limitations but that nevertheless provides a valid first-order approximation proving to be adequate for many purposes.

The SDG model, in the form to be described, is formulated for inputs that are expressed as functions of only a single variable, as described above. Although this is true also of the usual formulation for PSD techniques, the PSD model has been developed to a significant degree to cover the case of two-dimensional inputs, including the problem of spanwise-gradient effects. For example, Etkin[1] has discussed the extension of power spectral theory to the two-dimensional case, introducing in particular a four-point model of the aircraft.

Discrete Representation of Patterns of Fluctuation

We have referred, in Sec. I, to the importance of phase correlation as defined in the Fourier representation of turbulence. As a basis for introducing the significant phase correlations, the SDG method employs a signal representation in which patterns of fluctuation are modeled in terms of elementary ramp-shaped elements (Fig. 1b). Such an approach is consistent with the notion that the basic ingredient of pattern in time or space is change, or difference. On the other hand, in the frequency plane such a discrete signal comprises a set of phase-correlated Fourier components.

The dynamic systems with which we shall be concerned all have a high-pass characteristic; that is, they do not respond significantly to very-long-wavelength inputs. A corollary is that the response to a step input will ultimately decay to zero. Such a property is typical of the transient dynamic response to gusts of many quantities of practical interest, such as incremental aircraft normal acceleration, angular pitch rate, or associated structural loads. Peaks in response are thus closely associated with changes or increments in turbulence velocity rather than with the total amplitude of turbulence velocity itself, and it proves sufficient for many applications to model statistically the gradient of turbulence velocity rather than the overall magnitude. Thus, the SDG method represents non-Gaussian turbulence by means of discrete pulses in velocity gradient (Fig. 1a), corresponding to smooth-ramp elements in velocity itself (Fig. 1b), and introduces the statistical properties of turbulence in terms of the amplitude distributions and clustering properties of these elements.

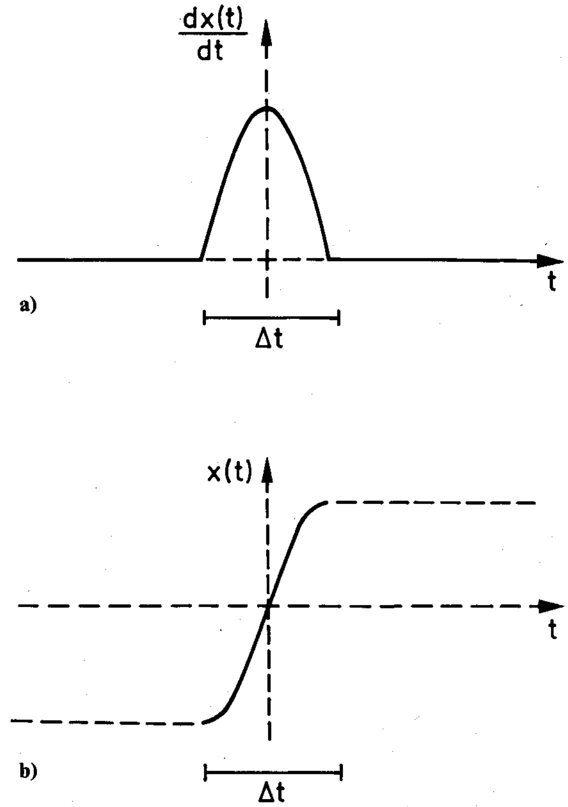

Fig. 1 Smooth ramp profile: a) localized pulse in signal gradient; b) smooth ramp signal.

Brillouin[19] has discussed the frequency content of limited-duration signals and shown them to be also band-limited, the duration Δt and bandwidth Δf being related by an "uncertainty relation":

$$\Delta t \cdot \Delta f \geq \tfrac{1}{2} \tag{1}$$

For a discrete pulse, such as that illustrated in Fig. 1a, the uncertainty relation takes the form[19]

$$\Delta t \cdot \Delta f \doteq \tfrac{1}{2} \tag{2}$$

(The exact value of the constant of proportionality depends on the precise definition of Δt and Δf and is not important for our purposes.)

The elements employed in the SDG theory are chosen to satisfy the further constraint of constant relative bandwidth, in the form

$$\Delta f / f_0 \doteq 1 \tag{3}$$

It follows from Eqs. (2) and (3) that

$$2\Delta t \doteq (\Delta f)^{-1} \doteq f_0^{-1} \tag{4}$$

Thus, Δt corresponds approximately to one half-cycle at the f_0. A localized concentration of signal gradient dx/dt on such a domain (Fig. 1a) occurs when the signal $x(t)$ itself takes the form of an increment over Δt (Fig. 1b).

Statistical Model for Elementary Fluctuations

We first discuss the statistical properties of the individual smooth ramps, regarded as independent entities. We shall employ a frozen-field (Taylor's) hypothesis to transform between the spatial structure of a $v(s)$ and the corresponding time-dependent function as encountered by an aircraft traversing the turbulence field. Moreover, we restrict attention to inertial-range turbulence, initially making the assumption that velocity components satisfy a condition of self-similarity[3,6,20] with similarity parameter $k = 1/3$. This is consistent with classical (Kolmogorov[21]) similarity theory for the inertial subrange. An extension of this basic theory to incorporate a more general fractal model of turbulence is presented in Sec. V.

We introduce a self-similar statistical model[3,6] of a turbulence-velocity component as an aggregate of smooth-ramp elements, as illustrated in Fig. 1, covering a range of H and w (Fig. 2), where $w = \Delta v$, $H = V\Delta t$, and Δt is the duration of the increment as experienced by a sensor traveling with mean airspeed V. The range of H is bounded at the upper end by the L (Fig. 2) and at the lower end by the limit (at a length of order 1-10 cm) imposed by the action of viscosity (not shown in Fig. 2).

From the w and the H we derive an additional intensity variable

$$u_k = w/H^k \tag{5}$$

which plays a central role in the specification of probability distributions.

For such a self-similar process a statistical model of elementary fluctuations is defined as follows. We let $N_{H,u_k}dH$ denote the average rate of occurrence (per unit distance s) of smooth-ramp elements with gradient distance in the range $(H, H + dH)$ and intensity greater than u_k. This combination of a density with respect to H and a cumulative distribution with respect to u_k is chosen for subsequent analytical convenience. Positive and negative increments will in general be combined into one family but may be modeled separately if it is desired to take explicit account of skewness. Conditions of self-similarity,[3,6] in the form of the statistical invariance of $v(s)$ with respect to a transformation in which the s and v axes are expanded by respective factors in the ratio $h:h^k$, imply that N_{H,u_k} takes the form

$$N_{H,u_k} = (a/H^2)F(u_k/b) \tag{6}$$

where a and b are constants and F is a dimensionless universal function, assumed to be monotonically decreasing with increasing u_k. The factor H^{-2} arises because the constant a is nondimensional and N_{H,u_k} has the dimensions of number per unit distance s and per unit increment dH. The a is referred to as the intermittency parameter; its value determines whether the fluctuations are densely packed or are relatively sparse. The b is referred to as the intensity parameter, since the effect of changing its numerical value is simply to alter the overall amplitude or intensity of the fluctuations.

The distribution of smooth-ramp elements [Eq. (6)] is expressed in terms of the function $F(u_k/b)$. As discussed in Ref. 6, the probability distributions of turbulence-velocity gradients and increments tend to be strongly non-Gaussian, even in situations where the overall velocity itself may take a distribution close to normal. As a result, the function F decays, with increasing u_k, more slowly than would be the case for a Gaussian process.

The method of system-response prediction to be described in Sec. IV involves the assumption of such a non-Gaussian distribution for increments, at least asymptotically for large u_k. Specifically, the universal function $F(u_k/b)$ [Eq. (6)] is assumed to take the asymptotic form for large u_k:

$$F(u_k/b) \equiv \exp[-(u_k/b)] \tag{7}$$

Evidence for the appropriate form for the amplitude distribution of turbulence-velocity increments has been discussed by Barndorff-Nielsen,[22] who proposed a parametric family of distributions, involving exponential functions, which are shown to agree well with measured data. Such a distribution had in fact been introduced much earlier, to model turbulence-velocity increments, in Ref. 3.

Combining Eqs. (5-7), it follows that the joint probability density with respect to H and w is proportional to $\{a/bH^{k+2}\}\exp\{-w/bH^k\}$. For large w the exponential term dominates, and increments satisfying the relationship $w \sim H^k$, or $u_k =$ const (Fig. 2) are asymptotically equiprobable.

Under such asymptotic conditions, for large u_k, the smooth-ramp elements in the family defined by Eqs. (6) and (7) will be assumed to be nonoverlapping in the space-wave number plane; they may overlap in space but only if they are separated with respect to wave number (or spatial frequency). In fact, in applying the model to predict system response, we will go further and make the stronger assumption that the more intense elements are either sufficiently well-separated in the space-wave number plane to have independent effects on large peaks in response, or, when they do have a significant interactive effect, that they may be associated into discrete patterns for which further statistical properties are introduced, as described in the following subsection.

Signal Patterns Consisting of Finite Clusters of Smooth-Ramp Elements

The time-frequency representations associated with several simple patterns built up from smooth-ramp components are shown in Fig. 3. The shaded regions correspond to the domains of significant energy concentration in the time-frequency plane [strictly concentrations of gradient energy $\int(dx/dt)^2\,dt$] associated with the respective smooth ramps. Note that the f scale has been compressed for illustrative convenience [with a linear scale the shaded regions would all have equal area, from Eq. (2)]. Examples of aircraft dynamic response for which these patterns provide "tuned gust" inputs (in a sense to be defined in Sec. IV) are described in Refs. 7 and 23.

A gust pattern comprising a cluster of n ramp elements, or n pattern, is defined in terms of a $(3n - 1)$-dimensional configuration vector

$$\boldsymbol{H}_n = \{H_1, H_2, \ldots, H_n, J_1, \ldots, J_{n-1}, \sigma_1, \ldots, \sigma_n\} \tag{8}$$

where the H_i are the gradient distances (Fig. 2) of the n-component ramps, the σ_i are discrete variables, equal to ± 1, which define the sign of the associated ramp (increasing or decreasing), and the variables $J_1, \ldots, J_{n-1}$ define the relative positions of the ramps. For example, for a two pattern comprising a nonoverlapping pair of ramps, as illustrated in Fig. 3a, $\sigma_1 = 1$, $\sigma_2 = -1$, and one variable, J_1, is required to define the spacing.

From the $\boldsymbol{H}_n$ a single variable H may be derived to represent the overall size or scale of the pattern. One obvious such measure would be $(H_1 \ldots H_n J_1 \ldots J_{n-1})^{1/2n-1}$. However, it is convenient, and in practice adequate, to employ the approximate measure

$$H = H(\boldsymbol{H}_n) = (H_1 H_2 \ldots H_n)^{1/n} \tag{9}$$

which depends only on the component gradient distances.

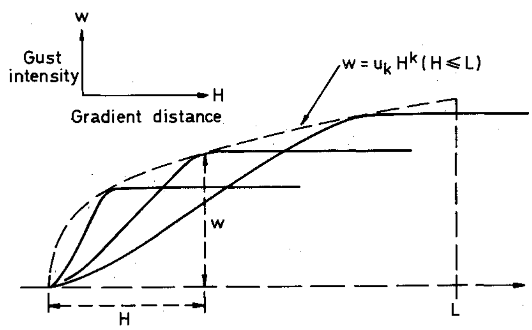

Fig. 2 Family of ramp gusts defined for $H \leq L$ by u_k.

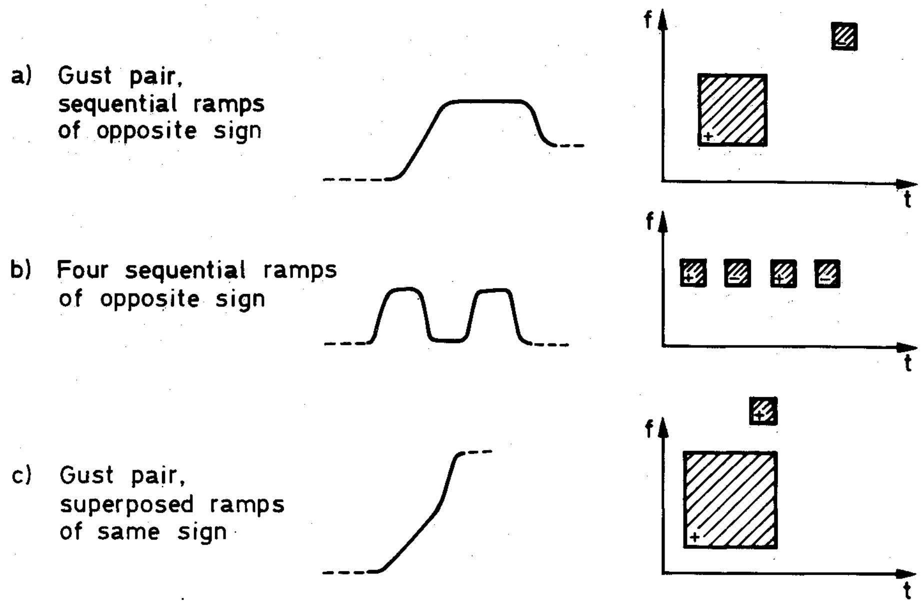

Fig. 3 Gust patterns and related time/frequency diagrams.

Furthermore, given any configuration H_n we denote by H_n^* the class of all patterns that can be generated from H_n by a scale-changing transformation, i.e., by simple enlargement or compression. Thus, any individual pattern may alternatively be defined by the pair of variables (H_n^*,H), which specify, respectively, the shape and scale.

In addition to the pattern shape and scale, the complete specification of an n pattern also includes the dimension of intensity. For a single elementary fluctuation, the intensity is defined by Eq. (5). As a basis for generalizing this equation, the hypothesis is introduced that the w_i of the individual components of an n pattern be constrained such that

$$w_1/H_1^k = w_2/H_2^k = \ldots = w_n/H_n^k \tag{10}$$

It follows from these equations that, viewed as independent fluctuations, the components (H_i,w_i) take equal values of the intensity variable [Eq. (5)].

Although the constraints imposed by Eq. (10) significantly reduce the number of degrees of freedom attached to an n pattern, it has been verified by computer simulation[7] that a sufficient number of degrees of freedom has in fact been retained in the input model for the purposes of predicting system response to a good approximation. Relaxation of these constraints [Eq. (10)] would lead to some degree of increased accuracy but at a substantial cost in terms of the additional complexity of implementation.

Statistical Model for Discrete-Gust Patterns

The statistical model for discrete-gust patterns, which comprise finite clusters of smooth-ramp components, is an extension of the model for isolated ramp gusts [Eq. (5-7)]. Specifically, the equation generalizing Eq. (6) takes the form

$$N_{H_n,u_k} = (a/H^{2n})F(u_k/b) \tag{11}$$

where $N_{H_n,u_k}dH_1 \ldots dH_n\, dJ_1 \ldots dJ_{n-1}$ denotes the rate of occurrence, per unit distance, of patterns with configuration vector in the neighborhood of H_n defined by the ranges $(H_1,H_1+\mathrm{d}H_1),\ldots,(J_1,J_1+\mathrm{d}J_1)\ \ldots$ and intensity variable greater than u_k. Generalizing Eq. (5), u_k is determined from the equation

$$u_k = \frac{w_i}{H_i^k p[H_n^*]}, \qquad 1 \le i \le n \tag{12}$$

Equation (12) incorporates Eq. (10) and also takes into account the effect of pattern shape on probability through the factor $p[H_n^*]$. For any given pattern, with H_n and w_i, the factor $p[H_n^*]$ determines u_k, from Eq. (12), and hence the rate of occurrence, from Eq. (11).

The statistical model for n patterns is completed by assuming the exponential form for F [Eq. (7)]. It then follows, as for isolated ramp gusts, that for patterns with large amplitudes w_i the exponential term dominates the probability density and n patterns satisfying the equation u_k = const are asymptotically equiprobable. To this approximation the factor $p[H_n^*]$ [Eq. (12)] has the effect of associating n patterns into families of equal probability, where the families are parameterized by u_k.

IV. Effects of Turbulence on System Dynamic Response

The SDG method is concerned predominantly with the occurrence of large peaks in system output. The statistical distribution of response peaks is expressed as an integral, over discrete input patterns, whose asymptotic form for large amplitudes is evaluated by the Laplace approximation. To this approximation, for any prescribed dynamic system, the large response peaks are associated with input patterns whose configuration H_n, as defined in Sec. III, lies in the neighborhood of a particular pattern that is matched, or tuned, to the system in question. The procedure for finding the tuned input pattern corresponding to a given system involves a variational problem in which the system peak response is maximized with respect to input patterns subject to a constraint related to their probability of occurrence. Alternatively, the procedure for finding the tuned input pattern may be expressed in an equivalent dual form in which the probability of occurrence of the input pattern is maximized subject to a constraint on response-peak amplitude.

Linear-System Response to Self-Similar Process Comprising Ramp Inputs Having Independent Effects

We first discuss the situation in which the input is a self-similar process comprising individual ramp gusts that are assumed to have completely independent effects on the amplitudes of large peaks in the system output and in which, moreover, the system is linear. In the following subsection we retain the assumption of self-similarity but extend the treatment to include more general patterns of excitation that are modeled as clusters of ramp elements having a strongly interactive effect on response peaks. We also extend the derivation to cover the case of systems with strongly nonlinear response characteristics.

The condition that individual ramp gusts have independent effects on large peaks in system response involves both the statistical characteristics of the input fluctuations and the system dynamic behavior. If the ramp gusts are sufficiently well-separated, there is no particular constraint on system behavior. As the separation between individual gusts is decreased, however, the possibility of an interactive effect arises that is dependent on the system "memory" as reflected in the relative damping of system-response modes and hence on the rate at which response fluctuations decay. In the case, for example, of a simple second-order oscillator whose damping is sufficiently high for the transient response to an isolated ramp gust to comprise simply a monotonic rise to a peak followed by immediate decay with relatively small overswing of the datum level, numerical simulation studies[7] show that the input may be modeled as an aggregate of ramp gusts having independent effects even in the case of so-called continuous turbulence. As the system damping ratio is steadily decreased, however, first pair interactions and subsequently higher-order interactions become significant.

In addition to its application to the response of well-damped dynamic systems, the analysis of the present section may also be applied to the response of broadband digital filters and thus provides the basis of a quantitative method for fitting statistical model to measured turbulence records. This topic is discussed, and examples are presented, in Refs. 24 and 25.

Consider first the peak response of a given linear system to a family of ramp gusts constrained to have unit intensity u_k [Eq. (5)]. The amplitude of a gust of duration H is thus given by $w = H^k$ (Fig. 2). For each value of H we denote the amplitude and epoch or phase of the largest response peak by the function pair $\{\gamma(H), \psi(H)\}$ (Fig. 4). The functions $\gamma(H)$ and $\psi(H)$ play a role in the present analysis closely analogous to that of the amplitude and phase of the system frequency-response function, where, however, we have used a (broadband) smooth ramp as input rather than a pure sine wave. Since the system is assumed to be linear, for an input consisting of a smooth ramp of arbitrary intensity u_k, we have

$$\gamma(H) = \text{peak response amplitude}/u_k \tag{13}$$

It is assumed that $\gamma(H)$ has a stationary maximum value (Fig. 5) within the interval $0<H<L$, where respective values at this maximum are denoted by

$$H = \bar{H}, \qquad \gamma(\bar{H}) = \bar{\gamma}, \qquad \psi(\bar{H}) = \bar{\psi} \tag{14}$$

Now consider the response of the system to an input process comprising a self-similar aggregate of ramp gusts as defined by Eq. (6), where the gusts of high intensity are assumed to be noninterfering in their effects on large peaks in response. Counting only the single largest peak in the response to any gust, let y denote an arbitrary level of amplitude and $n_{H,y}\,dH$ be the average rate of occurrence of response peaks with amplitude greater than or equal to y due to all gusts with duration in the range $(H, H + dH)$. It then follows from the definition of N_{H,u_k} and Eq. (6) that

$$n_{H,y} = N_{H,y/\gamma(H)} = \frac{a}{H^2} F\left\{\frac{y}{b\gamma(H)}\right\} \tag{15}$$

From the assumption of noninterference it follows that the average rate of occurrence n_y of peaks with amplitude greater than or equal to y is obtained, for large y, simply by integrating with respect to H. Assuming the exponential form for F [Eq. (7)], we thus have

$$n_y = \int_0^L n_{H,y}\,\mathrm{d}H = \int_0^L \frac{a}{H^2}\exp\left\{-\frac{y}{b\gamma(H)}\right\}\mathrm{d}H \tag{16}$$

Equation (16) is of the form

$$n_y = \int_0^L \phi(H)\exp\{yg(H)\}\mathrm{d}H$$

a standard integral that may be evaluated asymptotically, for large values of y, by the Laplace approximation to give

$$n_y = 2\phi(\xi)\left[\frac{-\pi}{2yg''(\xi)}\right]^{1/2}\exp\{yg(\xi)\} \tag{17}$$

provided $g(H)$ reaches its maximum at a stationary value, at which $g''(H)$ is continuous, when $H = \xi$, $0<\xi<L$.

In physical terms, although $g(H)$ may have a fairly "flat" maximum at $H = \xi$, the integrand factor $\exp\{gy(H)\}$ will have an increasingly sharp maximum at $H = \xi$ as $y \to \infty$. Thus, asymptotically the contribution to the integral from the neighborhood of $H = \xi$ dominates.

Identifying $g(H)$ with $-\{b\gamma(H)\}^{-1}$ and noting that $\xi = \bar{H}$, $\gamma'(\bar{H}) = 0$, and $\gamma''(\bar{H})<0$, Eq. (16) may be evaluated asymptoticaly to give

$$n_y = \frac{a}{\lambda\bar{H}}\left\{\frac{y}{b\gamma(\bar{H})}\right\}^{-1/2}\exp\left\{\frac{-y}{b\gamma(\bar{H})}\right\} \tag{18}$$

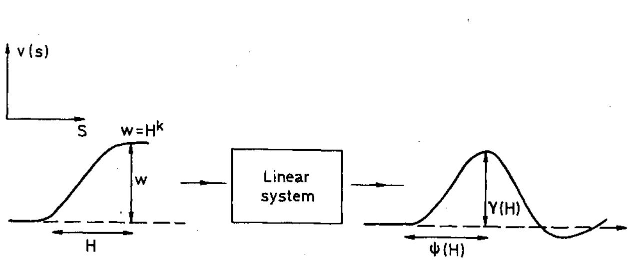

Fig. 4 Amplitude and phase functions for peak response to smooth-ramp input.

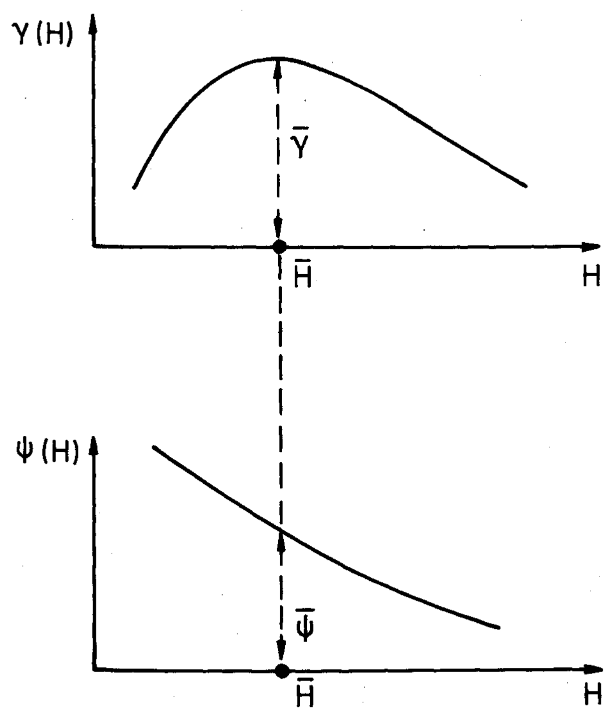

Fig. 5 Illustration of tuned-response parameters $\bar{\gamma}$ and $\bar{\psi}$.

where

$$\lambda = \bar{H}\left[\frac{-\gamma''(\bar{H})}{2\pi\gamma(\bar{H})}\right]^{1/2} \tag{19}$$

is a nondimensional factor representing the curvature of $\gamma(H)$ at its stationary maximum value.

Equation (18) shows that the gusts contributing significantly to n_y have gradient distance approximately equal to $\bar{H}$. Thus, to a first approximation a response peak of magnitude greater than y will be associated with a ramp gust of intensity u_k greater than $y/\gamma(\bar{H})$ [from Eq. (13)].

Suppose now that the gradient distances of gusts contributing to n_y come from a finite range $\bar{h}$. The average number of gusts per unit distance in this range having intensity greater than $y/\gamma(\bar{H})$ is approximately $N_{\bar{H},y/\gamma(\bar{H})}\bar{h}$. Thus, the number of response peaks per unit distance will be given approximately by

$$n_y \doteqdot N_{\bar{H},y/\gamma(\bar{H})}\bar{h}$$

that is,

$$n_y \doteqdot \frac{a\bar{h}}{\bar{H}^2}\exp\left\{-\frac{y}{b\gamma(\bar{H})}\right\} \tag{20}$$

from Eqs. (6) and (7). Comparing Eqs. (18) and (20), we see that if we take

$$\bar{h} = \frac{\bar{H}}{\lambda}\left\{\frac{y}{b\gamma(\bar{H})}\right\}^{-1/2} \tag{21}$$

we may (asymptotically) replace the approximate equality sign in Eq. (20) by an equality sign. Thus, the range of gradient distances contributing significantly to n_y is of order $y^{-1/2}$ and narrows with increasing y. This dependence of response on an increasingly narrow range of inputs at high intensities may be interpreted as a form of "transient resonance," to be generalized to more complex input patterns in the following subsection.

Of course, a question remains concerning the extent to which the asymptotic result [Eq. (18)] is applicable at levels of amplitude of practical interest. This issue may be approached by means of numerical simulation, using measured samples of turbulence and a variety of simulated dynamic systems. Examples supporting the validity of the method as a practical tool are presented in Refs. 7 and 10.

Finally we note that, according to Eq. (18), $\lambda\bar{H}n_y$ is a universal function of $y/\gamma(\bar{H})$ of the form $e^{-z}/z^{1/2}$. For practical applications we regard the existence of a functional relationship as the primary result, rather than the precise theoretical form of the function. In fact, empirical results indicate that the universal function may be represented equally effectively by a simple exponential of the form $\alpha e^{-z/\beta}$ over finite ranges of amplitude of practical interest. That is, peak-response counts based on empirical data[24,25] may be represented by the equation

$$\lambda\bar{H}n_y = \alpha\exp\{-y/[\beta\gamma(\bar{H})]\} \tag{22}$$

rather than by Eq. (18), and α and β may be related to a and b by matching functions as shown in Fig. 6. An adequate approximation for a wide range of amplitudes is given by

$$\alpha = 0.89a$$

$$\beta = b/1.12$$

In effect, having shown in Eq. (21) that $\bar{h}$ decreases asymptotically like $y^{-1/2}$ as y increases, in practical applications we generally employ a constant average value of $\bar{h}$ appropriate to the range of y of interest.

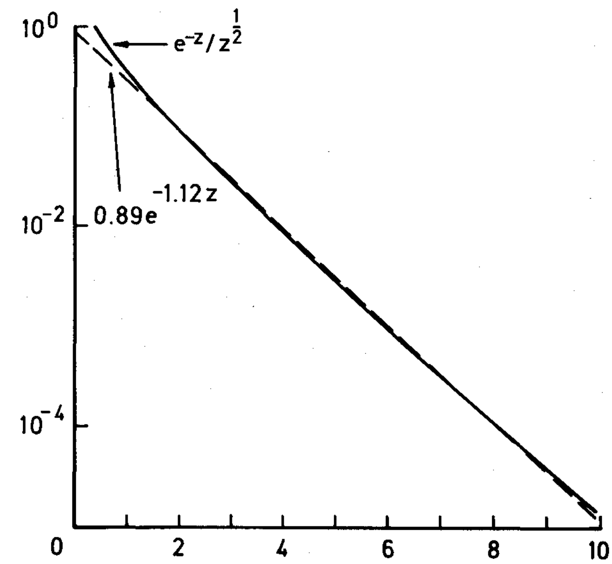

Fig. 6 Approximation to $e^{-z}/z^{1/2}$ by simple exponential function.

Response of Linear and Nonlinear Systems to a Self-Similar Process Comprising More Complex Gust Patterns

The analytical approach employed in the previous subsection may be extended to establish a statistical relationship between the occurrence of discrete input patterns of arbitrary complexity and associated large peaks in system-response in situations where the system may be linear or nonlinear. We present just an outline of this extension.

Suppose the excitation to comprise a self-similar family of discrete patterns $\{H_n, u_k\}$ with distribution determined by Eqs. (11) and (7), where n ranges over integer values. Counting just the single largest peak in the response to any one pattern, let $n_{H_n,y}dH_1\ldots dH_n dJ_1\ldots dJ_{n-1}$ denote the average rate of occurrence, per unit distance, of response peaks with amplitude greater than or equal to y, due to n patterns with configuration in the neighborhood $(H_1, H_1 + dH_1), \ldots, (J_1, J_1 + dJ_1), \ldots,$ of H_n.

For a given system, linear or nonlinear, and particular H_n, let $u_k(H_n, y)$ denote the value of the u_k required to produce a response peak of y. We assume $u_k(H_n, y)$ to be a monotonically increasing function of y. Then, from the definition of $N_{H_n u_k}$ and Eq. (11), it follows that

$$n_{H_n,y} = \frac{a}{H^{2n}}F\left\{\frac{u_k(H_n,y)}{b}\right\} \tag{23}$$

Again counting just the single largest peak and summing over n and configurations H_n [Eq. (8)], including permutations of the discrete sign variables σ_i, the total average rate of occurrence n_y of response peaks with amplitude greater than or equal to y is given by

$$n_y = \sum_{n,\sigma_i}\underbrace{\int\cdots\int}_{(2n-1)} n_{H_n,y}\,\mathrm{d}H_1\ldots\mathrm{d}J_1\ldots$$

$$= \sum_{n,\sigma_i}\underbrace{\int\cdots\int}_{(2n-1)}\frac{a}{H^{2n}}F\left\{\frac{u_k(H_n,y)}{b}\right\}\mathrm{d}H_1\ldots\mathrm{d}J_1\ldots$$

Assuming the exponential form [Eq. (7)] for F, we thus have

$$n_y = \sum_{n,\sigma_i} \underbrace{\int \cdots \int}_{(2n-1)} \frac{a}{H^{2n}} \exp\left\{-\frac{u_k(H_n,y)}{b}\right\} \mathrm{d}H_1 \ldots \mathrm{d}J_1 \ldots \tag{24}$$

By a generalization of Laplace's method [Eq. (17)] to more than one dimension, Eq. (24) may be evaluated to give an asymptotic expression for large values of y, analogous to that already obtained for independent ramp inputs [Eq. (22)]. The resulting equation[26,27] is

$$n_y = \frac{\alpha}{\lambda \bar{H}_m(y)} \exp\left\{-\frac{u_k(y)_m}{\beta}\right\} \tag{25}$$

The terms in this equation are obtained from a variational procedure in which y is prescribed successively at different levels, at each of which a search is made over H_n to find the global minimum value of $u_k(H_n,y)$. This minimum value, occuring when $n = m$ is denoted by $u_k(y)_m$, and the associated value of H is denoted by $\bar{H}_m(y)$. The λ is a pattern-sensitivity factor that generalizes the expression in Eq. (19) and α, β are adjusted values of a, b, as in Eq. (22). Strictly, λ should also be a function of y, but for practical applications the y variation has been found to be negligible. As is generally the case with variational methods involving a constraint, a dual method exists in which the roles of the variables are reversed. Specifically, in the dual method a succession of values for u_k is prescribed, for each of which H_n is varied to find the stationary maximum value

$$y = y(u_k)_m \tag{26}$$

of the peak amplitude $y(H_n,u_k)$. The associated equation for n_y, replacing Eq. (25), is then[26]

$$n_y = \frac{\alpha}{\lambda \bar{H}_m(u_k)} \exp\left\{-\frac{u_k}{\beta}\right\} \tag{27}$$

In this formulation, n_y is related to y implicitly, through Eqs. (26) and (27), u_k playing the role of an independent parameter.

In simulation studies it has been found that the pattern-sensitivity factor λ varies relatively little from system to system. In applications it may be replaced, in Eqs. (25) and (27), by an overall constant of order 0.3.

Particular Results for Linear Systems

For linear systems, the maximum response [Eq. (26)] becomes directly proportional to u_k, and the scale of the tuned pattern $\bar{H}_m$ [Eq. (27)] becomes independent of u_k. It is then possible to eliminate u_k from Eqs. (26) and (27), to give

$$\lambda \bar{H}_m n_y = \alpha \exp\{-[y/(\beta \bar{\gamma}_m)]\} \tag{28}$$

where $\bar{\gamma}_m$ is the maximum peak amplitude of response obtainable when the input patterns H_n are varied subject to the constraint $u_k = 1$. This result generalizes Eq. (22), derived previously for the case of single ramp inputs.

For linear systems, a further simplification arises in the implementation of the search for the worst-case tuned input. Using the principle of superposition, the maximum response to a pattern comprising n ramp components may be found by separately maximizing the n largest peaks in the response to a single ramp input, replacing the search in a space of several dimensions by n one-dimensional searches. The associated tuned input pattern $\bar{H}_n$, for any particular n, may subsequently be synthesized by locating the n individual tuned ramps, resulting from the n one-dimensional searches, relative to one another in space and sign such that the n associated response peaks reinforce one another. Details of this procedure, together with the results of computer-simulation studies providing confirmation of the predictive power of the method for linear systems, are presented in Ref. 7.

V. Scale-Dependent Intermittency

Although the classical Kolmogorov[21] (1941) universal similarity theory provides quite a good representation of high-Reynolds-number turbulence in the inertial subrange, it has more recently come to be accepted that it gives only a first approximation. Subsequent theoretical work by Kolmogorov,[28] Oboukhov,[29] and others explored the consequences of taking into account the spatial randomness of dissipation, not considered in the earlier theory. For our purposes the principal conclusion has been that, in contrast to the self-similarity of structure at different scales in the original Kolmogorov[21] theory, the spatial non-uniformity of dissipation is associated with systematically increasing intermittency with decreasing scale. Compared with the self-similar model, fluctuations become greater in magnitude, but more locally concentrated, as they cascade to smaller wavelengths. Experimental confirmation of such scale-dependent intermittency has been given by Van Atta and Park[30] and others.

Scale-dependent intermittency in the inertial range has been considered in detail by Mandelbrot,[13] who introduced the concept of the D of the support, or active volume, of turbulence, where $2 < D \leq 3$, and by Frisch et al.,[14] who formulated an explicit geometrical model of turbulence structure, the β-model, incorporating the parameter D. When $D = 3$, the β-model reduces to the classical Kolmogorov[21] theory. However, current theoretical estimates[13-15] put D in the range of 2.5–2.8. When $D < 3$, the fractional volume occupied by active turbulence steadily decreases with decreasing scale, implying the systematic compression of turbulence energy into reduced volume. This has the consequence of relatively increased intensity within the active volumes and is of considerable significance[31] for aircraft responding to the shorter wavelengths.

In the following discussion, we illustrate the way in which scale-dependent intermittency, characterized by the D, may be incorporated into SDG theory. Details will be given for the statistical model of isolated ramp gusts; the extension to more complex patterns is analogous.

In Eq. (6), intermittency is taken into account through the parameter a. Scale-dependent intermittency may be introduced into the model by replacing the constant a by a monotonically increasing function $a(H)$. Thus, Eq. (6) is replaced by

$$N_{H,u_k} = \frac{a(H)}{H^2} F\left(\frac{u_k}{b}\right) \tag{29}$$

At the same time, to allow for the associated increased intensity as turbulence is compressed on to reduced volumes, the k [Eq. (5)] must be decreased below the value (one-third) appropriate for the self-similar model. The specific form of $a(H)$ and new value for k may be determined from the β-model of Frisch et al.[14]

The key assumptions underlying the β-model are that, at scales of order 2^{-n}, only a fraction β^n (hence, the name of the model) of the total volume of space is significantly active, and that the complete statistics of velocity differences within such active regions of differing scales become identical under appropriate scaling of velocities. On the basis of the β-model hypotheses,[14,31] it follows that

$$a(H) \sim H^{3-D} \tag{30}$$

$$k = 1/3 - [(3 - D)/3] \tag{31}$$

The self-similar model is recovered as a special case by setting $D = 3$. Then $k = 1/3$ and, from Eq. (5), $u_k = w/H^{1/3}$. In contrast, taking the lower limit $D = 2.5$, we have $k = 1/6$ and $u_k = w/H^{1/6}$. As discussed in Sec. III, the condition $u_k = \text{const}$ is asymptotically a condition of constant probability. The alternative power laws $w \sim H^{1/3}$ and $w \sim H^{1/6}$ corresponding to the values $D = 3$ and $D = 2.5$ of the fractal dimension are shown in Fig. 7. The practical significance of varying the value of D is most pronounced for the very short gusts.

Subsequent to the formulation of the β-model, a more refined theoretical analysis[15] of the energy-cascade process in turbulent flow has indicated that, rather than taking a single constant value, the D is a function of the intensity of the fluctuation in turbulence velocity. This has been supported by an analysis[32] of measurements of continuous turbulence, which indicate that the D takes the value $D = 3$ at low intensities but tends asymptotically toward an extrapolated value of D of order 2.5 at the most extreme intensities. Implications of this result for possible future airworthiness requirements are mentioned in Sec. VII.

The effects of scale-dependent intermittency on the statistics of large peaks in system response are most simply illustrated for linear systems. Retaining the exponential form for the function F [Eq. (7)], but replacing Eq. (6) by Eq. (29), the average rate of occurrence n_y of response peaks greater than or equal to y is obtained by generalizing Eq. (16):

$$n_y = \int_0^L \frac{a(H)}{H^2} \exp\left\{-\frac{y}{b\gamma(H)}\right\} dH \tag{32}$$

where $\gamma(H)$ is defined by Eq. (13), u_k is given by Eqs. (5) and (31), and $a(H)$ takes the form of the product aH^{3-D}, from Eq. (30).

Applying the Laplace approximation, Eq. (32) may be evaluated asymptotically to give an expression analogous to that in Eq. (22):

$$\lambda(\bar{H})^{D-2} n_y = \alpha \exp\left\{\frac{-y}{\beta\gamma(\bar{H})}\right\} \tag{33}$$

where $\gamma(\bar{H})$ is now calculated using Eqs. (13), (5), and (31).

Results corresponding to Eq. (33) have been used to compare the effects on aircraft response of alternative choices of the parameter D in Ref. 31.

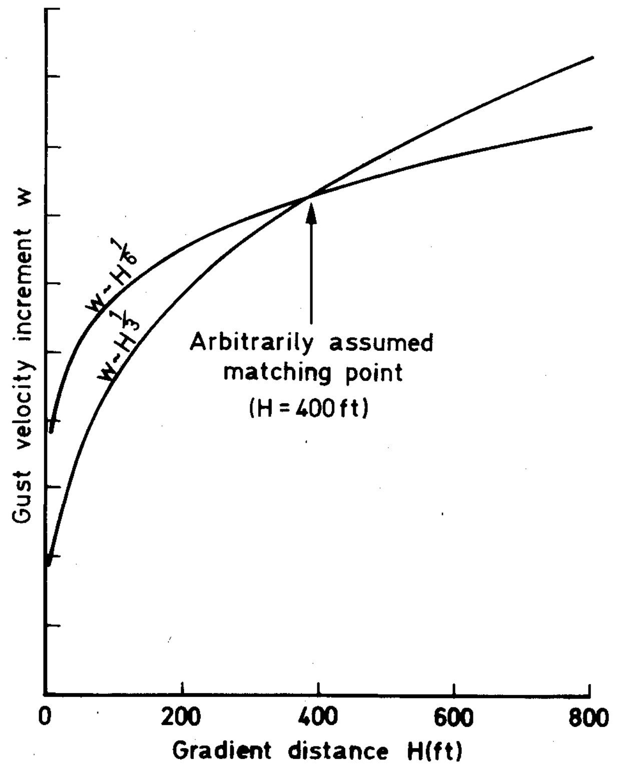

Fig. 7 Comparison of alternative $H^{1/3}$ and $H^{1/6}$ power laws.

VI. Effects of Nonlinear Aircraft Dynamics

It is a feature of the SDG method that linear and nonlinear systems can be assessed in a uniform manner. In each case, response evaluation is based on a worst-case analysis in which the maximum response to families of equiprobable input patterns is found. For nonlinear systems, however, the search for a worst case must be performed for several families of inputs, corresponding to different levels of intensity, whereas for linear systems only a single intensity needs to be considered.

In the application of the method to nonlinear systems, the relationship between the maximum system response and the associated input configuration (the tuned gust pattern) sometimes exhibits discontinuous or "jump" behavior. For instance, if the u_k [Eq. (12)] is increased in a continuous manner, the tuned gust pattern may change slowly and smoothly with u_k (in a linear problem it is independent of u_k); subsequently, however, a further small increase in u_k may lead to a sudden switch to a completely different pattern. Having established a new "branch," the variation may become smooth and gradual again. As seen in Sec. IV, denoting the peak amplitude of response to an input pattern H_n at intensity u_k by $y(H_n, u_k)$, we may envisage the function $y(H_n, u_k)$ as defining a class of "landscapes" whose shape depends on u_k. For any u_k, $y(u_k)_m$ [Eq. (26)] is the elevation of the highest hill. The above jump phenomenon arises when there exist two distinct local maxima (hills) in $y(H_n, u_k)$ and a value of u_k at which the two maxima are of equal magnitude but growing at different rates with respect to u_k. It follows from Eqs. (26) and (27) that the associated discontinuities in $(\mathrm{d}/\mathrm{d}u_k)y(u_k)_m$ and $\bar{H}_m(u_k)$ imply an associated discontinuity in the plot of n_y against y. Such possibilities emphasize the importance of investigating system behavior at the critical amplitudes of response and illustrate the weakness of extrapolating the function n_y from smaller values of y. A numerical study is described in Ref. 8 in which the statistical input/output relations predicted by the SDG method for various second-order systems having strong nonlinearities are shown to be in good agreement with the results of computer simulation.

For nonlinear systems, the principle of superposition is no longer applicable, and to find the maximum response to a pattern comprising n ramp components, a multidimensional search is required. Nevertheless, when interest lies primarily in the assessment of performance failures, which are relatively rare events, computational effort is being used in a more economical manner than in a standard Monte Carlo simulation in the sense that a systematic computer search to find the input pattern causing critical response is more efficient than simply waiting for the critical input pattern to come up at random. In a recent paper Noback[33] argued that the SDG search procedure may lead to prohibitively long simulation times. His reasoning, however, is based on the assumption that a uniform, exhaustive search of the input space is required. In fact, there is a substantial amount of literature[34,35] concerning closely analogous problems in statistical physics that shows how the steps in the simulation may efficiently combine a systematic (hill-climbing) search for a worst case with an element of randomization that prevents the search halting at subsidiary, rather than global, maxima.

VII. Specification of Numerical Parameters

It is a principal objective of the SDG method to cover, in a single unified approach, flight in patches of continuous turbulence and encounters with relatively isolated intense gusts. To conclude this paper we outline the methods used to determine numerical values for the parameters in the SDG model from measured data and review the current situation regarding the specification of these numerical values for the representation of continuous turbulence and relatively isolated gusts.

Continuous Turbulence

The SDG model is expressed, in Eqs. (5), (6), (11), and (12), in terms of the parameters k, a, b, and $p[H_n^*]$.

The value $k = 1/3$ for the similarity parameter[3,6] is consistent with the classical (Kolmogorov[21]) theory for self-similar turbulence in the inertial subrange, where the PSD is proportioned to $\kappa^{-5/3}$ and also with measurements[20,24,25,30] of turbulence over a wide range of amplitudes. However, we shall subsequently refer to theoretical and experimental evidence that this self-similar model requires modification at the higher gust intensities.

Restricting attention to the self-similar model, with $k = 1/3$, which is adequate for many practical applications such as ride quality,[23] handling-qualities studies, and perhaps also structural fatigue, there remains the choice of numerical values for a, b, and $p[H_n^*]$. The procedure whereby these parameters are determined from measured data depends on the theory, described in Sec. IV, for predicting the statistics of system response. For prescribed values of a, b, and $p[H_n^*]$, the theory [specifically, the equations for linear-system response, Eqs. (22) and (28)] can be used to predict the rate of occurrence of peaks in the response of a prescribed system, as a function of threshold-exceedance level. Conversely, regarding the system as a "window" through which the turbulence is observed, numerical values for the input-model parameters may be inferred by matching measured to predicted response statistics. When measured data are available in the form of records of turbulence velocity, rather than aircraft response, the same approach may be adopted using systems implemented as digital filters[24,25] whose response time histories are calculated by standard computer methods.

Numerical values for the a and b, corresponding to patches of turbulence extending over distances of the order of 5 miles, have been presented in Ref. 32 on the basis of measurements[25] of turbulence velocity made by a specially instrumented aircraft at altitudes below 1000 ft over a variety of terrains. Digital filters used in the parameter-matching process were based on numerical smoothing and differencing operations.[24] As a result of matching a self-similar model, with $k = 1/3$, to this data, a was found to vary from a value of 0.1 for very intermittent turbulence, in which bursts of high activity are interspersed with regions of much lower activity, to an upper limit of 0.8 for records in which the turbulence activity was continuous. The a and b were shown[32] to be independent variables but to be mutually related to the turbulence intensity as defined by the PSD. That is, a prescribed PSD function determines a unique relationship between a and b. Compared with a turbulence specification based just on PSD, the SDG model thus has an additional degree of freedom, in the form of intermittency.

Numerical values for the parameters $p[H_n^*]$ [Eq. (12)] have been obtained[7] by computer simulation of the response of oscillatory systems, covering a wide range of frequencies and damping ratios, using measured samples of continuous turbulence of moderate intensity. The results were further substantiated by predictions of the rate of occurrence of peaks in simulated aircraft loads at various points in the structure, with a full range of flexible modes. The particularly simple empirical result was obtained[7] that, for n patterns comprising nonoverlapping components of alternating sign, $p[H_n^*]$ depends to a good approximation only on the number n of components in the pattern. In particular, writing $p[H_n^*] = p_n$,

$$p_1 = 1.0, \qquad p_2 = 0.85, \qquad p_4 = 0.60, \qquad p_8 = 0.40 \tag{34}$$

These values have subsequently been shown[16,17] to be very close to the values that define the relative amplitudes of differently shaped patterns for a prescribed level of probability in a Gaussian process. A consequence is that for continuous turbulence (which may be strongly non-Gaussian) the ratios of the responses of different aircraft are approximately the same as if the input were Gaussian. This provides one interpretation of the fact that, for continuous turbulence, alternatives to Gaussian probability can be accommodated in a PSD analysis and implies that the results of an SDG analysis and a PSD analysis will be approximately equivalent.[7,9]

Relatively Isolated Gusts

The values of p_n given in Eqs. (34) were obtained from computer-simulation studies[7] using inputs in the form of measured samples of continuous turbulence. In subsequent work,[10] however, by means of a modified technique involving numerical simulation of the response of digitally implemented oscillators, using data in the form of measured samples of normal acceleration recorded during routine flying by civil aircraft, values of p_n were derived for gust encounters of sufficient intensity to produce a "special event" in which the aircraft response was at least 0.75 g. Compared with Eqs. (34), which corresponds to continuous turbulence, the p_n for $n > 1$, associated with these high-intensity events, were found[10] to take numerically smaller values. Translated into rates of occurrence for a given amplitude, the ratio of the rate of occurrence of patterns of higher complexity ($n = 2,4,8$) to that of single ramp gusts ($n = 1$) was found to be substantially smaller for high-intensity gust encounters than for continuous turbulence. In other words, the very intense fluctuations tend to occur predominantly as gusts that are relatively isolated. For this class of high-intensity events, a divergence thus occurs between the results of an SDG analysis and the results of a PSD analysis. The fact that very intense fluctuations in air velocity tend to occur predominantly as gusts that are relatively isolated has been known for a long time, but it was only when data from operational flying were fitted to an SDG representation that this trend was quantified in terms of statistical parameters directly applicable to aircraft gust-loads prediction.

A further difference between the values of the numerical parameters required to represent continuous turbulence on the one hand and relatively isolated intense gusts on the other concerns the k. Whereas empirical studies of extensive patches of turbulence support[32] the value $k = 1/3$ as a good overall descriptor, the tails of the distributions, corresponding to the most severe gusts, are better described[32] by a reduced value closer to $k = 1/6$. As discussed in Sec. V, these results are compatible with a multifractal model of turbulence[15] in which D reduces from a value of 3 at low amplitudes to a value approaching 2.5 at the highest intensities.

In Ref. 36 it was suggested that consideration should be given to the use of the value $k = 1/6$ for scaling extreme atmospheric gusts for the purpose of limit-load certification, retaining the value $k = 1/3$ for continuous turbulence of lower intensity. Whereas there is only a limited amount of data relevant to the choice of k for isolated extreme gusts, an analysis[37] of the data available up to 1979, although not conclusive, showed some support for the value $k = 1/6$. As is apparent from Fig. 7, the choice of the value of the k is of greatest significance at the shorter wavelengths. When it is taken into account that k is a property of some types of active control system on modern technology aircraft to reduce the gust-gradient distances corresponding to the predominant aircraft response,[23] the validation, or otherwise, of the proposal[36] to use the value $k = 1/6$ for the representation of extreme gusts in future airworthiness requirements remains a central topic for debate and future research.

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